

Monotonicity formulae, vanishing theorems and some geometric applications

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Abstract. Using the stress energy tensor, we establish some monotonicity formulae for vector bundle-valued p -forms satisfying the conservation law, provided that the base Riemannian (resp. Kähler) manifolds possess some real (resp. complex) p -exhaustion functions. Vanishing theorems follow immediately from the monotonicity formulae under suitable growth conditions on the energy of the p -forms. As an application, we establish a monotonicity formula for the Ricci form of a Kähler manifold of constant scalar curvature and then get a growth condition to derive the Ricci flatness of the Kähler manifold. In particular, when the curvature does not change sign, the Kähler manifold is isometrically biholomorphic to C^m . Another application is to deduce the monotonicity formulae for volumes of minimal submanifolds in some outer spaces with suitable exhaustion functions. In this way, we recapture the classical volume monotonicity formulae of minimal submanifolds in Euclidean spaces. We also apply the vanishing theorems to Bernstein type problem of submanifolds in Euclidean spaces with parallel mean curvature. In particular, we may obtain Bernstein type results for minimal submanifolds, especially for minimal real Kähler submanifolds under weaker conditions.

1 Introduction

In 1980, Baird and Eells [4] introduced the stress-energy tensor for maps between Riemannian manifolds, which unifies various results on harmonic maps. Following [4], Sealey [35] introduced the stress-energy tensor for p -forms with values in vector bundles and established some vanishing theorems for harmonic p -forms. Since then, the stress-energy tensors have become a useful tool for investigating the energy behaviour of vector bundle valued p -forms in various problems. Recently the authors

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in [15] presented a unified method to establish monotonicity formulae and vanishing theorems for vector-bundle valued p -forms satisfying conservation laws by means of the stress-energy tensors of various energy functionals in geometry and physics. Their method is based on a fundamental integral formula derived from the stress-energy tensor. Since the stress-energy tensor is a 2-tensor field, one gets a 1-form by contracting it with a vector field. The integral formula linked naturally to the conservation law follows directly from the divergence of the 1-form. In [15], the author mainly used the distance function on the base Riemannian manifold to construct a suitable vector field. The Hessian of the distance function appears naturally in the integral formula. Consequently they used Hessian comparison theorems and coarea formula to obtain those results.

The similar integral formula technique was also used by [13, 38, 39] to investigate harmonic maps between Kähler manifolds. Some special exhaustion functions on the domain Kähler manifolds were used to establish the monotonicity formulae of energy by [39] and the monotonicity formulae of the partial energies by [13] for the harmonic maps. Consequently some Liouville type results and holomorphicity results were obtained by [13] and [39] respectively. These results are actually equivalent to the vanishing of some 1-forms with values in the pull-back vector bundles.

In this paper, we shall establish monotonicity formulae and vanishing results for vector bundle valued p -forms satisfying the conservation laws by means of the stress-energy tensors too. Following [13], we hope to use some exhaustion functions instead of distance functions to construct suitable vector fields in the integral formulae derived by the stress-energy tensors. For these aims, we introduced the concepts of real (resp. complex) p -exhaustion functions on Riemannian manifolds (resp. Kähler manifolds) (see their definition in §2). Under some conditions on the radial curvature of a Riemannian manifold (resp. a Kähler manifold), the distance function actually becomes a real p -exhaustion function (resp. a complex p -exhaustion function). The existence of a real p -exhaustion function on the base manifold enables us to establish a monotonicity formula for a vector bundle valued p -form satisfying the conservation law and the growth order of the energy of the p -form is determined only by the exhaustion function (see Theorem 3.1). Inspired by [13], we introduce the notion of J -invariant p -forms, which includes the differential of a pluriconformal map, the partial differentials of a map between Kähler manifolds, the second fundamental of a Kähler submanifold and the Ricci form of a Kähler manifold, etc. It turns out that a complex p -exhaustion function is more suitable to estimate the growth order of the energy of a J -invariant p -form. As a result, the existence of a complex p -exhaustion function enables us to establish a monotonicity formula for a J -invariant p -form satisfying the conservation law too (see Theorem 3.4). Besides establishing the global monotonicity formulae, we also consider some conditions which allow us to establish the monotonicity formulae outside a compact subset of the base manifolds. Such a kind of monotonicity formulae is useful in some applications. Clearly vanishing the-

orems follow immediately from these monotonicity formulae under suitable growth conditions.

As an application of the above mentioned results, we establish a monotonicity formula for the Ricci form of a Kähler manifold of constant scalar curvature and then get a growth condition to derive the Ricci flatness of the Kähler manifold. Due to the fact that a Ricci form is a J -invariant 2-form, we derive these results on a Kähler manifold which possesses a complex 2-exhaustion function. Furthermore, if the radial curvature of M does not change sign, then the Ricci flatness implies that M is isometrically biholomorphic to C^m . In this way, we get some uniformization results. On the other hand, if we take $p = 0$ and consider a nonzero constant function in the monotonicity formulae, we may get monotonicity formulae for volumes of minimal submanifolds in some outer spaces with suitable exhaustion functions (see Theorem 4.1 and Theorem 4.2). In particular, we recapture the classical volume monotonicity formulae of minimal submanifolds in Euclidean spaces.

The benefits of using more general exhaustion functions instead of the distance functions in establishing monotonicity and vanishing results are that one may not only relax the curvature conditions on the domain manifolds but also has more choices for constructing suitable vector fields in the integral formula. As an application, we shall investigate Bernstein type problem for submanifolds in Euclidean spaces with parallel mean curvature. This is still an active project in recent years (see, for examples, [11, 20, 30, 32, 36, 37, 44, 46, 47] and the references therein). Recall that the Gauss maps of these submanifolds are harmonic. Hence the differentials of the Gauss maps satisfy the conservation law by a result of Baird-Eells [4]. It is natural to apply the previous vanishing theorems to establish Liouville theorems of the Gauss maps, which are equivalent to Bernstein type results for the submanifolds. We show that the extrinsic distance functions of the submanifolds become real 1-exhaustion functions if the second fundamental forms satisfy some decay conditions. Consequently we obtain some Bernstein type theorems for submanifolds with parallel mean curvature under certain decay conditions on the second fundamental forms and growth conditions on the integrals of the squared norms of the second fundamental forms. When the submanifolds are minimal, the Bernstein type results may be established under weaker conditions. Finally we consider the Bernstein problem for minimal submanifolds with finite total scalar curvature. Most results on this topic are obtained under the assumption that the minimal submanifolds have some stability. For example, the authors in [36] proved that any stable complete minimal hypersurface with finite total scalar curvature is a hyperplane. We refer the reader to [17, 24, 30, 31, 37, 42, 43] for recent progress on this topic and references therein. According to our technique, we may establish some Bernstein theorems for minimal submanifolds with finite total scalar curvature by assuming some boundary condition of a compact sublevel set of the extrinsic exhaustion distance function (see Corollary 5.3, Corollary 5.4 and Remark 5.4), instead of assuming the stable condition. In particular, we prove that a

minimal real Kähler submanifolds with finite total scalar curvature must be an affine plane.

2 Stress energy tensors and exhaustion functions

A smooth map $f : M \rightarrow N$ between two Riemannian manifolds is said to be a harmonic map if it is a critical point of the following energy functional:

$$E(f) = \frac{1}{2} \int_M |df|^2 dv$$

with respect to any compactly supported variation.

In [4], Baird-Eells introduced the stress-energy tensor S_f associated with $E(f)$ as follows

$$S_f = \frac{|df|^2}{2} g - df \odot df,$$

where $df \odot df \in \Gamma(T^*M \otimes T^*M)$ is a symmetric tensor defined by

$$(df \odot df)(X, Y) = \langle df(X), df(Y) \rangle.$$

Then they proved that a harmonic map satisfies the conservation law, that is, $\text{div} S_f = 0$.

Let (M, g) be a Riemannian manifold and $\xi : E \rightarrow M$ be a smooth Riemannian vector bundle over M with a metric compatible connection ∇^E . Set $A^p(\xi) = \Gamma(\Lambda^p T^*M \otimes E)$ the space of smooth p -forms on M with values in the vector bundle $\xi : E \rightarrow M$. The exterior covariant differentiation $d^\nabla : A^p(\xi) \rightarrow A^{p+1}(\xi)$ relative to the connection ∇^E is defined by (cf. [16])

$$(d^\nabla \omega)(X_1, \dots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} (\nabla_{X_i} \omega)(X_1, \dots, \widehat{X_i}, \dots, X_{p+1}).$$

The codifferential operator $\delta^\nabla : A^p(\xi) \rightarrow A^{p-1}(\xi)$ characterized as the adjoint of d^∇ is defined by

$$(\delta^\nabla \omega)(X_1, \dots, X_{p-1}) = -\sum_i (\nabla_{e_i} \omega)(e_i, X_1, \dots, X_{p-1}).$$

For $\omega \in A^p(\xi)$, we define the energy functional of ω as follows:

$$E(\omega) = \frac{1}{2} \int_M |\omega|^2 dv_g.$$

Its stress-energy tensor is:

$$(2.1) \quad S_\omega(X, Y) = \frac{|\omega|^2}{2} g(X, Y) - (\omega \odot \omega)(X, Y),$$

where $\omega \odot \omega \in \Gamma(A^p(\xi) \otimes A^p(\xi))$ is a symmetric tensor defined by

$$(\omega \odot \omega)(X, Y) = \langle i_X \omega, i_Y \omega \rangle.$$

Here $i_X \omega \in A^{p-1}(\xi)$ denotes the interior product by $X \in TM$. Notice that, if $p=0$, i.e., $\omega \in \Gamma(\xi)$, $i_X \omega = 0$ then (2.1) becomes

$$(2.2) \quad S_\omega(X, Y) = \frac{|\omega|^2}{2} g(X, Y).$$

For a 2-tensor field $T \in \Gamma(T^*M \otimes T^*M)$, its divergence $\text{div} T \in \Gamma(T^*M)$ is defined by

$$(\text{div} T)(X) = \sum_i (\nabla_{e_i} T)(e_i, X),$$

where $\{e_i\}$ is an orthonormal basis of TM . The divergence of S_ω is given by (cf. [3, 35, 45])

$$(2.3) \quad (\text{div} S_\omega)(X) = \langle \delta^\nabla \omega, i_X \omega \rangle + \langle i_X d^\nabla \omega, \omega \rangle.$$

For a vector field X on M , its dual one form θ_X is given by

$$\theta_X(Y) = g(X, Y) \quad \forall X \in TM.$$

The covariant derivative of θ_X gives a 2-tensor field $\nabla \theta_X$:

$$(\nabla \theta_X)(Y, Z) = (\nabla_Z \theta_X)(Y) = g(\nabla_Z X, Y), \quad \forall Y, Z \in TM.$$

If $X = \nabla \psi$ is the gradient of some smooth function ψ on M , then $\theta_X = d\psi$ and $\nabla \theta_X = \text{Hess}(\psi)$.

For any vector field X on M , a direct computation yields (cf. [45] or Lemma 2.4 of [15]):

$$(2.4) \quad \text{div}(i_X S_\omega) = \langle S_\omega, \nabla \theta_X \rangle + (\text{div} S_\omega)(X).$$

Let D be any bounded domain of M with C^1 boundary. By (2.4) and using the divergence theorem, we immediately have the following integral formula (see [15, 45]):

$$(2.5) \quad \int_{\partial D} S_\omega(X, \nu) ds_g = \int_D [\langle S_\omega, \nabla \theta_X \rangle + (\text{div} S_\omega)(X)] dv_g.$$

where ν is the unit outward normal vector field along ∂D . In particular, if ω satisfies the conservation law, i.e. $\text{div} S_\omega = 0$, then

$$(2.6) \quad \int_{\partial D} S_\omega(X, \nu) ds_g = \int_D \langle S_\omega, \nabla \theta_X \rangle dv_g.$$

To apply the above integral formula, we introduce some special exhaustion functions. Let (M^m, g) ($m \geq 2$) be a m -dimensional Riemannian manifold and let Φ be a Lipschitz continuous function on M^m satisfying the following conditions:

(2.7) $\Phi \geq 0$ and Φ is an exhaustion function of M , i.e., each sublevel set $B_\Phi(t) := \{\Phi < t\}$ is relatively compact in M for $t \geq 0$;

(2.8) $\Psi = \Phi^2$ is of class C^∞ and Ψ has only discrete critical points;

(2.9) The constant $k_1 = \inf_{x \in M} \{\lambda_1(x) + \lambda_2(x) + \cdots + \lambda_m(x) - 2p\lambda_m(x)\} > 0$, where p is a nonnegative integer, $\lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_m(x)$ are the eigenvalues of $Hess(\Psi)$, and the constant $k_2 = \sup_{x \in M} |\nabla \Phi|^2$ is finite.

When (M, g) is a Kähler manifold, we sometimes need the following notion of exhaustion functions, which is defined by the complex Hessian instead of the (real) Hessian of a function.

Let Φ be a Lipschitz continuous function on M^m ($\dim_C M = m > 1$) satisfying (2.7), (2.8) and the following condition:

(2.10) The constant $\bar{k}_1 = \inf_{x \in M} \{\sum_{i=1}^m \epsilon_i(x) - p\epsilon_m(x)\}$ is positive where $\epsilon_1 \leq \epsilon_2 \leq \cdots \leq \epsilon_m$ are the eigenvalues of the complex Hessian $H(\Psi) = (\Psi_{i\bar{j}})$. The constant $\bar{k}_2 = \sup_{x \in M} |\nabla \Phi|^2$ is finite. Set

$$(2.11) \quad \lambda = \frac{k_1}{2k_2}, \quad \bar{\lambda} = \frac{\bar{k}_1}{\bar{k}_2}.$$

The function Φ with the properties (2.7), (2.8) and (2.9) (resp. (2.10)) will be called real (resp. complex) p -exhaustion function in this paper. Notice that a complex 1-exhaustion function is just the special exhaustion function discussed in [13, 39].

Now we give some examples of real and complex p -exhaustion functions.

Example 2.1. For R^m , we take $\Psi = \Phi^2 = \sum_{i=1}^m \frac{(x_i)^2}{a_i}$, where a_i are any positive constants satisfying $1 \leq a_1 \leq a_2 \leq \cdots \leq a_m$, then Φ is a smooth exhaustion function of R^m , and Ψ has only discrete critical points. For $x \in R^n - \{0\}$, we have

$$\begin{aligned} \nabla \Phi(x) &= \sum_{i=1}^m \frac{x_i/a_i}{\sqrt{\sum_{i=1}^m (x_i)^2/a_i}} \frac{\partial}{\partial x_i}, \\ Hess(\Psi)(x) &= \sum_{i=1}^m \frac{2}{a_i} dx_i \otimes dx_i. \end{aligned}$$

So

$$k_1 = \frac{2}{a_1} + \frac{2}{a_2} + \cdots + \frac{2}{a_m} - \frac{4p}{a_1}, \quad k_2 = \sup_{x \in R^m} |\nabla \Phi|^2 \leq 1.$$

Example 2.2. For C^m , we take $\Psi = \Phi^2 = \frac{|z_1|^2}{a_1} + \cdots + \frac{|z_m|^2}{a_m}$, where a_i are any positive constants satisfying $1 \leq a_1 \leq a_2 \leq \cdots \leq a_m$. A direct computation gives

$$\bar{k}_1 = \frac{2}{a_1} + \frac{2}{a_2} + \cdots + \frac{2}{a_m} - \frac{2p}{a_1}, \quad \bar{k}_2 = \sup_{x \in C^m} |\nabla \Phi|^2 \leq 1.$$

Clearly we may choose suitable a_i such that k_1 (resp. \bar{k}_1) > 0 . Therefore the function Φ in Example 2.1 (resp. in Example 2.2) becomes a real (resp. complex) p -exhaustion function.

Example 2.3. For a complete submanifold in Euclidean space R^N , we shall show in §5 that the restriction of the extrinsic distance function becomes a real 1-exhaustion (or even p -exhaustion) function if the second fundamental form of the submanifold satisfies some decay condition (see the proof of Theorem 5.1 or Remark 5.1).

Example 2.4. Let $i : M^m \rightarrow C^N$ be an m -dimensional submanifold and $F : C^N \rightarrow R$ be a smooth function on C^N . Then the composition formula for Hessian of maps gives

$$Hess(F \circ i)(\eta_j, \bar{\eta}_k) = (HessF)(\eta_j, \bar{\eta}_k) + dF(B(\eta_j, \bar{\eta}_k)),$$

where B denotes the second fundamental form of M and $\{\eta_j = \frac{1}{\sqrt{2}}(e_j - iJe_j)\}_{j=1}^m$ is any unitary frame tangent to M . Since M is a complex submanifold, we have

$$Hess(F \circ i)(\eta_j, \bar{\eta}_k) = (HessF)(\eta_j, \bar{\eta}_k).$$

Now let $F = \|z\|^2 = \sum_{A=1}^N |z_A|^2$ and set $\Psi = \Phi^2 = F \circ i$. Clearly the eigenvalues of the complex Hessian $H(\Psi)$ are $\epsilon_1 = \dots = \epsilon_m = 2$. If $p < m$, then

$$\bar{k}_1 = \inf_{x \in M} \left\{ \sum_{i=1}^m \epsilon_i - p\epsilon_m \right\} = 2(m - p) > 0.$$

Therefore Φ is a complex p -exhaustion function with growth order $\bar{\lambda} \geq 2(m - p)$, because $\bar{k}_2 \leq 1$. Recall that every Stein manifold M^m can be realized as a closed submanifold of C^N by a proper holomorphic map $\psi : M^m \rightarrow C^N$. Thus a Stein manifold M admits a complex p -exhaustion function $\Psi = \Phi^2$ with $\Phi = \psi^*(\|z\|)$ and $\bar{\lambda} \geq 2(m - p)$. Notice that the complex Hessian is computed with respect to the induced metric by ψ . It is well known that every closed complex submanifold of a Stein manifold is a Stein manifold too. Hence Stein manifolds provide us many examples of Kähler manifolds which poss complex p -exhaustion functions.

We now give some complete Riemannian manifolds (resp. Kähler manifolds) whose distance functions are real (resp. complex) p -exhaustion functions.

Lemma 2.1. *Let $(M^m, g)(m \geq 2)$ be an m -dimensional complete Riemannian manifold with a pole x_0 and let r be the distance function relative to x_0 . Assume that there exist two positive functions $h_1(r)$ and $h_2(r)$ such that*

$$h_1(r)[g - dr \otimes dr] \leq Hess(r) \leq h_2(r)[g - dr \otimes dr]$$

in the sence of quadratic forms, then

$$(2.12) \quad \sum_{i=1}^m \lambda_i - 2p\lambda_m \geq \begin{cases} 2 + 2(m-1)rh_1(r) - 4prh_2(r) & \text{if } rh_2(r) \geq 1, \\ 2 + 2(m-1)rh_1(r) - 4p & \text{if } rh_2(r) < 1, \end{cases}$$

where $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m$ are eigenvalues of $Hess(r^2)$. In particular, if M is a Kähler manifold of complex dimension m , then

$$(2.13) \quad \sum_{i=1}^m \epsilon_i - p\epsilon_m \geq \begin{cases} 1 + (2m-1)rh_1(r) - 2prh_2(r) & \text{if } rh_1(r) \geq 1, \\ 1 + (2m-1)rh_1(r) - p[1 + rh_2(r)] & \text{if } rh_2(r) < 1. \end{cases}$$

where $\epsilon_1 \leq \epsilon_2 \leq \dots \leq \epsilon_m$ are eigenvalues of the complex Hessian $H(r^2) = ((r^2)_{i\bar{j}})$.

Proof. It is known that $Hess(r^2)$ is given by

$$Hess(r^2) = 2dr \otimes dr + 2rHess(r).$$

By the assumption, we have

$$(2.14) \quad Hess(r^2) \leq 2dr \otimes dr + 2rh_2(r)[g - dr \otimes dr].$$

If $rh_2(r) \geq 1$ (resp. < 1), then $\lambda_m \leq 2rh_2(r)$ (resp. $= 2$). Therefore

$$\begin{aligned} \sum_{i=1}^m \lambda_i - 2p\lambda_m &\geq 2 + 2(m-1)rh_1(r) - 4prh_2(r) \\ &\quad (\text{resp. } 2 + 2(m-1)rh_1(r) - 4p). \end{aligned}$$

This proves the first assertion.

Now suppose M is a Kähler manifold of complex dimension m . Clearly we have (cf. also Lemma 4.4 of [13])

$$(2.15) \quad \sum_{i=1}^m \epsilon_i \geq 1 + (2m-1)rh_1(r).$$

By the assumption, we get

$$(2.16) \quad \epsilon_m \leq \begin{cases} 2rh_2(r) & \text{if } rh_2(r) \geq 1, \\ 1 + rh_2(r) & \text{if } rh_2(r) < 1. \end{cases}$$

In conclusion, we derive the following

$$\sum_{i=1}^m \epsilon_i - p\epsilon_m \geq \begin{cases} 1 + (2m-1)rh_1(r) - 2prh_2(r) & \text{if } rh_1(r) \geq 1, \\ 1 + (2m-1)rh_1(r) - p[1 + rh_2(r)] & \text{if } rh_2(r) < 1. \end{cases}$$

■

Remark 2.1. (1) The authors in [15] considered the stress energy tensors associated with more general functional, that is, the F -energy. If we take $F(t) = t$, the proof of Lemma 4.2 in [15] essentially gives the lower bounds of $\sum_{i=1}^m \lambda_i - 2p\lambda_m$ under the assumption $rh_2(r) \geq 1$. (2) For $p = 1$, it is clear that the estimation (2.13) is independent of the upper bound of $Hess(r)$, which was also pointed out in [13].

Lemma 2.2. *Let (M, g) be a complete Riemannian manifold with a pole x_0 and let r be the distance function relative to x_0 . Denote by K_r the radial curvature of M .*

(i) *If $-\alpha^2 \leq K_r \leq -\beta^2$ with $\alpha > 0$, $\beta > 0$, then*

$$\beta \coth(\beta r)[g - dr \otimes dr] \leq \text{Hess}(r) \leq \alpha \coth(\alpha r)[g - dr \otimes dr].$$

(ii) *If $-\frac{A}{(1+r^2)^{1+\epsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\epsilon}}$ with $\epsilon > 0$, $A \geq 0$, $0 \leq B < 2\epsilon$, then*

$$\frac{1 - \frac{B}{2\epsilon}}{r}[g - dr \otimes dr] \leq \text{Hess}(r) \leq \frac{e^{\frac{A}{2\epsilon}}}{r}[g - dr \otimes dr].$$

(iii) *If $-\frac{a^2}{1+r^2} \leq K_r \leq \frac{b^2}{1+r^2}$ with $a \geq 0$, $b^2 \in [0, 1/4]$, then*

$$\frac{1 + \sqrt{1 - 4b^2}}{2r}[g - dr \otimes dr] \leq \text{Hess}(r) \leq \frac{1 + \sqrt{1 + 4a^2}}{2r}[g - dr \otimes dr].$$

Proof. The case (i) is standard (cf. [18]). The case (ii) is discussed in [15]. For (iii), we refer the reader to [18, 22, 30]. ■

In [15], the authors proved that the distance function r of the case (i) (resp. case (ii)) in Lemma 2.2 is a real p -exhaustion function by choosing suitable α and β (resp. A and B). For the case (iii), that is, the curvature of M has quadratic decay, we have immediately from (2.12) the following:

Lemma 2.3. *Let (M, g) be a complete Riemannian manifold with a pole x_0 . If $-\frac{a^2}{1+r^2} \leq K_r \leq \frac{b^2}{1+r^2}$ with $a \geq 0$, $b^2 \in [0, 1/4]$ and $2 + (m - 2)(1 + \sqrt{1 - 4b^2}) - (2p - 1)(1 + \sqrt{1 + 4a^2}) > 0$, then $r(x)$ is a real p -exhaustion function with*

$$(2.17) \quad \lambda = \frac{2 + (m - 1)(1 + \sqrt{1 - 4b^2}) - 2p(1 + \sqrt{1 + 4a^2})}{2}.$$

Lemma 2.4. *Let M^m be a complete Kähler manifold of complex dimension m with a pole. Assume that the radial curvature of M satisfies one of the following three conditions:*

- (i) $-\alpha^2 \leq K_r \leq -\beta^2$ with $\alpha > 0$, $\beta > 0$ and $(2m - 1)\beta - 2p\alpha > 0$;
- (ii) $-\frac{A}{(1+r^2)^{1+\epsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\epsilon}}$ with $\epsilon > 0$, $A \geq 0$, $0 \leq B < 2\epsilon$ and $1 + (2m - 1)[1 - \frac{B}{2\epsilon}] - 2pe^{A/2\epsilon} > 0$;
- (iii) $-\frac{a^2}{1+r^2} \leq K_r \leq \frac{b^2}{1+r^2}$ with $a \geq 0$, $b^2 \in [0, 1/4]$ and $2 + (2m - 1)[1 + \sqrt{1 - 4b^2}] - 2p[1 + \sqrt{1 + 4a^2}] > 0$.

Then r is a complex p -exhaustion function with growth order

$$(2.18) \quad \bar{\lambda} = \begin{cases} 2[m - p\frac{\alpha}{\beta}] & \text{if } K_r \text{ satisfies (i),} \\ 1 + (2m - 1)(1 - \frac{B}{2\epsilon}) - 2pe^{A/2\epsilon} & \text{if } K_r \text{ satisfies (ii),} \\ 1 + \frac{(2m-1)[1+\sqrt{1-4b^2}]}{2} - p[1 + \sqrt{1 + 4a^2}] & \text{if } K_r \text{ satisfies (iii).} \end{cases}$$

Proof. If K_r satisfies (i), then by Lemma 2.2 and the increasing function $\beta r \coth \beta r \rightarrow 1$ as $r \rightarrow 0$, we meet the case (i) of (2.13). Since $\beta r \coth(\beta r) > 1$ for $r > 0$, $\frac{\coth(\alpha r)}{\coth(\beta r)} < 1$ for $0 < \beta < \alpha$, we get

$$\begin{aligned}
\sum_{i=1}^m \epsilon_i - p\epsilon_m &\geq 1 + (2m - 1)\beta r \coth(\beta r) - 2p\alpha r \coth(\alpha r) \\
&= 1 + \beta r \coth(\beta r) [(2m - 1) - 2p \frac{\alpha r \coth(\alpha r)}{\beta r \coth(\beta r)}] \\
&\geq 1 + (2m - 1) - 2p \frac{\alpha}{\beta} \\
&= 2[m - p \frac{\alpha}{\beta}],
\end{aligned}$$

provided that $(2m - 1)\beta - 2p\alpha \geq 0$.

If K_r satisfies (ii) (resp. (iii)), it follows from Lemma 2.2 that we can estimate $\sum_{i=1}^m \epsilon_i - p\epsilon_m$ by the second case of (2.13). ■

Remark 2.2. When M is a Kähler manifold of complex dimension m (i.e., real dimension $2m$), the distance function r may be both a real $2p$ -exhaustion function and a complex p -exhaustion function under suitable radial curvature conditions listed above. In general, the growth order $\bar{\lambda}$ is larger than the growth order λ .

3 Monotonicity formulae and Vanishing results

Theorem 3.1. Let (M, g) be an m -dimensional complete Riemannian manifold with a real p -exhaustion function Φ and let $\xi : E \rightarrow M$ be a Riemannian vector bundle on M . If $\omega \in A^p(\xi)$ satisfies the conservation law, that is, $\text{div} S_\omega = 0$, then

$$(3.1) \quad \frac{1}{\rho_1^\lambda} \int_{B_\Phi(\rho_1)} |\omega|^2 dv \leq \frac{1}{\rho_2^\lambda} \int_{B_\Phi(\rho_2)} |\omega|^2 dv$$

for any $0 < \rho_1 \leq \rho_2$, where λ is given by $\lambda = k_1/2k_2$.

Proof. Take $\Psi = \Phi^2$, $X = \frac{1}{2} \nabla \Psi$. Obviously $(\nabla \Psi)|_{\partial B_\Phi(t)}$ is an outward normal vector field along $\partial B_\Phi(t)$ for a regular value $t > 0$ of Φ . Thus $\nabla \Psi = \sigma(x)\nu$ on $\partial B_\Phi(t)$ with $\sigma(x) > 0$ for each point $x \in \partial B_\Phi(t)$, where ν denotes the unit outward normal vector field of $\partial B_\Phi(t)$. Take an orthonormal basis $\{e_i\}_{i=1,2,\dots,m}$ which diagonalize the $Hess(\Psi)$, then

$$\begin{aligned}
\langle S_\omega, \nabla \theta_X \rangle &= \frac{1}{2} \sum_{i,j=1}^m S_\omega(e_i, e_j) Hess(\Psi)(e_i, e_j) \\
&= \frac{1}{4} \sum_{i,j=1}^m |\omega|^2 Hess(\Psi)(e_i, e_j) \delta_{ij}
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \sum_{i,j=1}^m (\omega \odot \omega)(e_i, e_j) Hess(\Psi)(e_i, e_j) \\
& \geq \frac{|\omega|^2}{4} \inf_{x \in M} \{(\lambda_1 + \dots + \lambda_m) - 2p\lambda_m\} \\
(3.2) \quad & \geq \frac{1}{4} k_1 |\omega|^2.
\end{aligned}$$

By the definition of S_ω , we have

$$\begin{aligned}
S_\omega(X, \nu) &= \frac{|\omega|^2}{2} g(X, \nu) - (\omega \odot \omega)(X, \nu) \\
&= \frac{1}{2} r |\omega|^2 g(\nabla \Phi, \nu) - \frac{1}{2} \sigma(x) |i_\nu \omega|^2 \\
(3.3) \quad &\leq \frac{\sqrt{k_2}}{2} r |\omega|^2 \quad \text{on } \partial B_\Phi(r).
\end{aligned}$$

Since ω satisfies the conservation law, then by (2.6), (3.2) and (3.3), we can get

$$\frac{1}{2} k_1 \int_{B_\Phi(r)} |\omega|^2 dv \leq \sqrt{k_2} r \int_{\partial B_\Phi(r)} |\omega|^2 dv.$$

By the definition of k_2 and the co-area formula, we have

$$\begin{aligned}
r \int_{\partial B_\Phi(r)} |\omega|^2 ds &\leq r \sqrt{k_2} \int_{\partial B_\Phi(r)} \frac{|\omega|^2}{|\nabla \Phi|} ds \\
&= r \sqrt{k_2} \frac{d}{dr} \int_{B_\Phi(r)} |\omega|^2 dv.
\end{aligned}$$

Denote by $\lambda = k_1/2k_2$, then

$$\lambda \int_{B_\Phi(r)} |\omega|^2 dv \leq r \frac{d}{dr} \int_{B_\Phi(r)} |\omega|^2 dv,$$

thus

$$\frac{\frac{d}{dr} \int_{B_\Phi(r)} |\omega|^2 dv}{\int_{B_\Phi(r)} |\omega|^2 dv} \geq \frac{\lambda}{r}.$$

Integrating the above formula on $[\rho_1, \rho_2]$, we can get

$$\frac{1}{\rho_1^\lambda} \int_{B_\Phi(\rho_1)} |\omega|^2 dv \leq \frac{1}{\rho_2^\lambda} \int_{B_\Phi(\rho_2)} |\omega|^2 dv.$$

■

Sometimes it is useful to establish monotonicity formulae outside a compact subset on M . Under an extra condition on the p -form ω , we may establish such a kind of monotonicity formulae.

Theorem 3.2. *Let (M, g) be an m -dimensional complete Riemannian manifold and let $\xi : E \rightarrow M$ be a Riemannian vector bundle on M , $\omega \in A^p(\xi)$. Suppose Φ is an exhaustion function on M and is a real p -exhaustion function on $M \setminus B_\Phi(R_0)$ for some $R_0 > 0$. Set*

$$\begin{aligned} k_1(R_0) &= \inf_{x \in M \setminus B_\Phi(R_0)} \{\lambda_1(x) + \lambda_2(x) + \cdots + \lambda_m(x) - 2p\lambda_m(x)\}, \\ k_2(R_0) &= \sup_{x \in M \setminus B_\Phi(R_0)} |\nabla \Phi|^2. \end{aligned}$$

If ω satisfies the conservation law, and

$$(3.4) \quad \frac{|\omega|^2}{2} \geq |i_\nu \omega|^2 \quad \text{on } \partial B_\Phi(R_0),$$

where ν denotes the unit outward normal vector field of $\partial B_\Phi(R_0)$, then

$$(3.5) \quad \frac{1}{\rho_1^{\lambda(R_0)}} \int_{B_\Phi(\rho_1) \setminus B_\Phi(R_0)} |\omega|^2 dv \leq \frac{1}{\rho_2^{\lambda(R_0)}} \int_{B_\Phi(\rho_2) \setminus B_\Phi(R_0)} |\omega|^2 dv$$

for any $R_0 < \rho_1 \leq \rho_2$, where $\lambda(R_0) = k_1(R_0)/2k_2(R_0)$.

Proof. Take $X = \frac{1}{2}\nabla \Phi^2 = \Phi \nabla \Phi$. By the definition of S_ω and (3.4), we get

$$\begin{aligned} S_\omega(\nu, \nu) &= \frac{|\omega|^2}{2} g(\nu, \nu) - (\omega \odot \omega)(\nu, \nu) \\ &= \frac{1}{2} |\omega|^2 - |i_\nu \omega|^2 \\ &\geq 0. \end{aligned}$$

For any $r > R_0$, we set $D = B_\Phi(r) \setminus B_\Phi(R_0)$, by applying the integral formula (2.6) on D and using (3.2), (3.3), we have

$$\begin{aligned} \frac{1}{4} k_1(R_0) \int_{B_\Phi(r) \setminus B_\Phi(R_0)} |\omega|^2 dv &\leq \int_{\partial B_\Phi(r)} S_\omega(X, \nu) ds - \int_{\partial B_\Phi(R_0)} S_\omega(X, \nu) ds \\ &= \int_{\partial B_\Phi(r)} S_\omega(X, \nu) ds - R_0 \int_{\partial B_\Phi(R_0)} S_\omega(\nu, \nu) ds \\ (3.6) \quad &\leq \frac{\sqrt{k_2(R_0)}}{2} r \int_{\partial B_\Phi(r)} |\omega|^2 ds, \end{aligned}$$

and by the co-area formula

$$\begin{aligned} r \int_{\partial B_\Phi(r)} |\omega|^2 ds &\leq r \sqrt{k_2(R_0)} \int_{\partial B_\Phi(r)} \frac{|\omega|^2}{|\nabla \Phi|} ds \\ (3.7) \quad &\leq r \sqrt{k_2(R_0)} \frac{d}{dr} \int_{B_\Phi(r) \setminus B_\Phi(R_0)} |\omega|^2 dv. \end{aligned}$$

Then (3.6) and (3.7) yield

$$\lambda(R_0) \int_{B_\Phi(r) \setminus B_\Phi(R_0)} |\omega|^2 dv \leq r \frac{d}{dr} \int_{B_\Phi(r) \setminus B_\Phi(R_0)} |\omega|^2 dv,$$

which implies that

$$(3.8) \quad \frac{d}{dr} \{ r^{-\lambda(R_0)} \int_{B_\Phi(r) \setminus B_\Phi(R_0)} |\omega|^2 dv \} \geq 0.$$

By integrating (3.8) on $[\rho_1, \rho_2]$, we can get the formula (3.5). ■

Suppose M is a submanifold described in Example 2.3 on page 7. Then we may establish the monotonicity formulae for forms on M . Next, Lemma 2.3 and Theorem 3.1 imply the following:

Theorem 3.3. *Let (M, g) be an m -dimensional complete Riemannian manifold with a pole x_0 . Let $\xi : E \rightarrow M$ be a Riemannian vector bundle on M and $\omega \in A^p(\xi)$. Assume that the radial curvature $-\frac{a^2}{1+r^2} \leq K_r \leq \frac{b^2}{1+r^2}$ with $a \geq 0$, $b^2 \in [0, 1/4]$ and $2 + (m-1)(1 + \sqrt{1-4b^2}) - 2p(1 + \sqrt{1+4a^2}) > 0$. If ω satisfies the conservation law, then*

$$\frac{1}{\rho_1^\lambda} \int_{B_{\rho_1}(x_0)} |\omega|^2 dv \leq \frac{1}{\rho_2^\lambda} \int_{B_{\rho_2}(x_0)} |\omega|^2 dv$$

for any $0 < \rho_1 \leq \rho_2$, where λ is given by

$$(3.9) \quad \lambda = \frac{2 + (m-1)(1 + \sqrt{1-4b^2}) - 2p(1 + \sqrt{1+4a^2})}{2}.$$

Definition 3.1. *A p -form $\omega \in A^p(\xi)$ is called J -invariant (or pluriconformal) if ω satisfies :*

$$(\omega \odot \omega)(JX, JY) = (\omega \odot \omega)(X, Y)$$

for $\forall X, Y \in T_x M$.

Now we consider J -invariant p -forms on Kähler manifolds. The work in [13] suggests that a complex p -exhaustion function is suitable to estimate the growth order of a J -invariant p -form which satisfies the conservation law.

Theorem 3.4. *Let M be a complete Kähler manifold which posses a complex p -exhaustion function Φ . Let $\xi : E \rightarrow M$ be a Riemannian vector bundle on M . If $\omega \in A^p(\xi)$ is J -invariant and satisfies the conservation law, then*

$$\frac{1}{\rho_1^{\bar{\lambda}}} \int_{B_\Phi(\rho_1)} |\omega|^2 dv \leq \frac{1}{\rho_2^{\bar{\lambda}}} \int_{B_\Phi(\rho_2)} |\omega|^2 dv$$

for any $0 < \rho_1 \leq \rho_2$, where $\bar{\lambda} = \bar{k}_1 / \bar{k}_2$.

Proof. Take $X = \frac{1}{2}\nabla\Phi^2 = \Phi\nabla\Phi$. Obviously $(\nabla\Psi)|_{\partial B_\Phi(t)}$ is an outward normal vector field along $\partial B_\Phi(t)$ for a regular value $t > 0$ of Φ . Thus $\nabla\Psi = \psi(x)\nu$ on $\partial B_\Phi(t)$ with $\psi(x) > 0$ for each point $x \in \partial B_\Phi(t)$, where ν denotes the unit outward normal vector field of $\partial B_\Phi(t)$.

$$\begin{aligned}
S_\omega(X, \nu) &= \frac{|\omega|^2}{2} \langle X, \nu \rangle_g - \langle i_X \omega, i_\nu \omega \rangle_g \\
&= t \frac{|\omega|^2}{2} \langle \nabla\Phi, \nu \rangle_g - \psi |i_\nu \omega|_g^2 \\
(3.10) \quad &\leq \frac{t\sqrt{k_2}}{2} |\omega|^2
\end{aligned}$$

on $\partial B_\Phi(t)$ and

$$\begin{aligned}
\langle S_\omega, \nabla\theta_X \rangle &= \frac{1}{2} \langle S_\omega, \text{Hess}(\Psi) \rangle \\
(3.11) \quad &= \frac{1}{2} \left[\frac{|\omega|^2}{2} \Delta_g \Psi - \langle \omega \odot \omega, \text{Hess}(\Psi) \rangle \right],
\end{aligned}$$

where Δ_g denote the Laplace-Beltrami operator on M . We choose a unitary basis $\{\eta_i = (e_i - iJe_i)/\sqrt{2}\}_{i=1, \dots, m}$ at a point $p \in B_\Phi(t)$ such that

$$\text{Hess}(\Psi)(\eta_i, \bar{\eta}_j(p)) = \epsilon_i \delta_{ij},$$

which is equivalent to

$$\begin{aligned}
\text{Hess}(\Psi)(e_i, e_j) + \text{Hess}(\Psi)(Je_i, Je_j) &= 2\epsilon_i \delta_{ij}, \\
(3.12) \quad \text{Hess}(\Psi)(e_i, Je_j) - \text{Hess}(\Psi)(Je_i, e_j) &= 0.
\end{aligned}$$

Obviously $\{e_i, Je_i\}_{i=1, \dots, m}$ is an orthonormal basis. Then

$$(3.13) \quad \Delta_g \Psi = 2 \sum_{i=1}^m \epsilon_i.$$

Since ω is J -invariant, we have

$$\begin{aligned}
(3.14) \quad &\langle \omega \odot \omega, \text{Hess}(\Psi) \rangle \\
&= \sum_{i,j} [\langle i_{e_i} \omega, i_{e_j} \omega \rangle \text{Hess}(\Psi)(e_i, e_j) + \langle i_{Je_i} \omega, i_{Je_j} \omega \rangle \text{Hess}(\Psi)(Je_i, Je_j) \\
&\quad + \langle i_{e_i} \omega, i_{Je_j} \omega \rangle \text{Hess}(\Psi)(e_i, Je_j) + \langle i_{Je_i} \omega, i_{e_j} \omega \rangle \text{Hess}(\Psi)(Je_i, e_j)] \\
&= \sum_{i,j} \langle i_{e_i} \omega, i_{e_j} \omega \rangle [\text{Hess}(\Psi)(e_i, e_j) + \text{Hess}(\Psi)(Je_i, Je_j)] \\
&\quad + \langle i_{e_i} \omega, i_{Je_j} \omega \rangle [\text{Hess}(\Psi)(e_i, Je_j) - \text{Hess}(\Psi)(Je_i, e_j)] \\
&= 2 \sum_{i,j} |i_{e_i} \omega|^2 \epsilon_i.
\end{aligned}$$

From (3.11), (3.13) and (3.14), we obtain

$$\begin{aligned}
\langle S_\omega, \nabla \theta_X \rangle &= \frac{1}{2} \left[\frac{|\omega|^2}{2} (2 \sum_{i=1}^m \epsilon_i) - 2 \sum_{i=1}^m |i_{e_i} \omega|^2 \epsilon_i \right] \\
&\geq \frac{1}{2} \left[|\omega|^2 \sum_{i=1}^m \epsilon_i - 2 \sum_{i=1}^m |i_{e_i} \omega|^2 \epsilon_m \right] \\
&= \frac{1}{2} \left[|\omega|^2 \sum_{i=1}^m \epsilon_i - p |\omega|^2 \epsilon_m \right] \\
&= \frac{1}{2} |\omega|^2 (\sum_{i=1}^m \epsilon_i - p \epsilon_m) \\
(3.15) \quad &\geq \frac{1}{2} \bar{k}_1 |\omega|^2,
\end{aligned}$$

where we have used the fact that $2 \sum_i |i_{e_i} \omega|^2 = p |\omega|^2$.

Since $\text{div} S_\omega = 0$, it follows from (2.6), (3.10) and (3.15) that

$$\frac{\sqrt{\bar{k}_2}}{2} t \int_{\partial B_\Phi(t)} |\omega|^2 ds \geq \frac{\bar{k}_1}{2} \int_{B_\Phi(t)} |\omega|^2 dv,$$

and the coarea formula yields

$$\frac{d}{dt} \int_{B_\Phi(t)} |\omega|^2 dv \geq \frac{1}{\sqrt{\bar{k}_2}} \int_{\partial B_\Phi(t)} |\omega|^2 dv.$$

Hence

$$(3.16) \quad \frac{\frac{d}{dt} \int_{B_\Phi(t)} |\omega|^2 dv}{\int_{B_\Phi(t)} |\omega|^2 dv} \geq \frac{\bar{\lambda}}{t}.$$

This theorem follows immediately from integrating (3.16) on $[\rho_1, \rho_2]$. ■

Theorem 3.5. *Let M be a complete Kähler manifold. Suppose Φ is an exhaustion function on M and is a complex 1-exhaustion function on $M \setminus B_\Phi(R_0)$ for some $R_0 > 0$. Set*

$$\begin{aligned}
\bar{k}_1(R_0) &= \inf_{x \in M \setminus B_\Phi(R_0)} \left\{ \sum_{i=1}^m \epsilon_i(x) - \epsilon_m(x) \right\}, \\
\bar{k}_2(R_0) &= \sup_{x \in M \setminus B_\Phi(R_0)} |\nabla \Phi|^2.
\end{aligned}$$

If $\omega \in A^1(\xi)$ is J -invariant and satisfies the conservation law, then

$$\frac{1}{\rho_1^{\bar{\lambda}(R_0)}} \int_{B_\Phi(\rho_1) \setminus B_\Phi(R_0)} |\omega|^2 dv \leq \frac{1}{\rho_2^{\bar{\lambda}(R_0)}} \int_{B_\Phi(\rho_2) \setminus B_\Phi(R_0)} |\omega|^2 dv$$

for any $R_0 < \rho_1 \leq \rho_2$, where $\bar{\lambda}(R_0) = \bar{k}_1(R_0)/\bar{k}_2(R_0)$.

Proof. Take $\Psi = \Phi^2$, $X = \frac{1}{2}\nabla\Psi$. For $\omega \in A^1(\xi)$, by the J -invariant of ω , we have

$$\begin{aligned}
S_\omega(\nu, \nu) &= \frac{|\omega|^2}{2}g(\nu, \nu) - (\omega \odot \omega)(\nu, \nu) \\
&\geq \frac{1}{2}[\langle \omega(\nu), \omega(\nu) \rangle + \langle \omega(J\nu), \omega(J\nu) \rangle] - \langle \omega(\nu), \omega(\nu) \rangle \\
(3.17) \quad &\geq 0.
\end{aligned}$$

For any $R > R_0$, we set $D = B_\Phi(R) \setminus B_\Phi(R_0)$, by applying the integral formula(2.6) on D and arguing in a similar way as in Theorem 3.2 and Theorem 3.4, considering (3.17), we can get the proof of this theorem. ■

From Example 2.4 and Theorem 3.4, we immediately have the following.

Theorem 3.6. *Let M^m be a complex m -dimensional Stein manifold with metric and exhaustion function Φ as in Example 2.4. Suppose $\xi : E \rightarrow M$ is a Riemannian vector bundle on M and $\omega \in A^p(\xi)$ with $p < m$ is J -invariant. If ω satisfies the conservation law, then*

$$\frac{1}{\rho_1^{2(m-p)}} \int_{B_\Phi(\rho_1)} |\omega|^2 dv \leq \frac{1}{\rho_2^{2(m-p)}} \int_{B_\Phi(\rho_2)} |\omega|^2 dv$$

for any $0 < \rho_1 \leq \rho_2$.

Theorem 3.7. *Let M^m ($\dim_{\mathbb{C}} M = m$) be a complete Kähler manifold with a pole x_0 . Let $\xi : E \rightarrow M$ be a Riemannian vector bundle on M . Assume that the radial curvature K_r of M satisfies one of the following three conditions:*

- (i) $-\alpha^2 \leq K_r \leq -\beta^2$ with $\alpha > 0$, $\beta > 0$ and $(2m-1)\beta - 2p\alpha \geq 0$;
- (ii) $-\frac{A}{(1+r^2)^{1+\epsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\epsilon}}$ with $\epsilon > 0$, $A \geq 0$, $0 \leq B < 2\epsilon$ and $1 + (2m-1)[1 - \frac{B}{2\epsilon}] - 2pe^{A/2\epsilon} > 0$;
- (iii) $-\frac{a^2}{1+r^2} \leq K_r \leq \frac{b^2}{1+r^2}$ with $a \geq 0$, $b^2 \in [0, 1/4]$ and $2 + (2m-1)[1 + \sqrt{1-4b^2}] - 2p[1 + \sqrt{1+4a^2}] > 0$.

If $\omega \in A^p(\xi)$ is J -invariant and satisfies the conservation law, then

$$\frac{1}{\rho_1^{\bar{\lambda}}} \int_{B_{\rho_1}(x_0)} |\omega|^2 dv \leq \frac{1}{\rho_2^{\bar{\lambda}}} \int_{B_{\rho_2}(x_0)} |\omega|^2 dv$$

for any $0 < \rho_1 \leq \rho_2$, where

$$(3.18) \quad \bar{\lambda} = \begin{cases} 2[m - p\frac{\alpha}{\beta}] & \text{if } K_r \text{ satisfies (i),} \\ 1 + (2m-1)(1 - \frac{B}{2\epsilon}) - 2pe^{A/2\epsilon} & \text{if } K_r \text{ satisfies (ii),} \\ 1 + \frac{(2m-1)[1+\sqrt{1-4b^2}]}{2} - p[1 + \sqrt{1+4a^2}] & \text{if } K_r \text{ satisfies (iii).} \end{cases}$$

Proof. It is a direct proof by Lemma 2.4 and Theorem 3.4. ■

Now we give some concrete examples of J -invariant forms which satisfy the conservation law.

Example 3.1. Let $f : (M, g, J) \rightarrow N$ be a smooth map from a Kähler manifold. We say that f is pluriconformal if $\langle df(JX), df(JY) \rangle = \langle df(X), df(Y) \rangle$. Hence the differential df of a pluriconformal harmonic map is a J -invariant 1-form with value in $f^{-1}TN$ which satisfies the conservation law.

Example 3.2. Let $f : (M, g, J) \rightarrow (N, h, J')$ be a smooth map between two Kähler manifolds. Define two 1-forms $\sigma, \tau \in A^1(f^{-1}TN)$

$$\sigma(X) = \frac{df(X) + J'df(JX)}{2}, \quad \tau(X) = \frac{df(X) - J'df(JX)}{2}$$

for any $X \in TM$. Then σ and τ are J -invariant. Furthermore, σ and τ satisfy the conservation laws if f is pluriharmonic map (cf. [13]).

Example 3.3. Let $\varphi : M \rightarrow N$ be a complex submanifold of a Kähler manifold. It is known that the second fundamental form B of M satisfies

$$(3.19) \quad B(JX, Y) = JB(X, Y).$$

If we regard $B \in A^1(T^*M \otimes \varphi^{-1}N)$, then (3.19) implies that B is a J -invariant 1-form with value in $T^*M \otimes \varphi^{-1}N$. When N is a complex space form, B satisfies the following Codazzi equation

$$(3.20) \quad (\nabla_X B)(Y, Z) = (\nabla_Y B)(X, Z).$$

Then (3.20) is equivalent to $d^\nabla B = 0$. Since a complex submanifold is automatically minimal, (3.20) yields $\delta^\nabla B = 0$ too. It follows from (2.3) that B satisfies the conservation law.

Example 3.4. The Ricci form Ω_{ric} of a Kähler manifold (M^m, g, J) is defined by $\Omega_{ric}(X, Y) = Ric(JX, Y)$. A basic property of Ω_{ric} is that $\Omega_{ric}(JX, JY) = \Omega_{ric}(X, Y)$. So Ω_{ric} is a J -invariant 2-form in the sense of Definition 3.1. On the other hand, we know that Ω_{ric} is closed and

$$(3.21) \quad \delta\Omega_{ric} = -\frac{1}{2}JdS,$$

where S denotes the scalar curvature of M (cf. [26]). By (3.21) and (2.3), we see that if M^m has constant scalar curvature, then its Ricci form Ω_{ric} satisfies the conservation law.

In [13], the author established some monotonicity formulae for the energy or partial energies of harmonic maps, pluriconformal harmonic maps and pluriharmonic maps by considering df , σ and τ . Consequently some Liouville results and holomorphicity results were obtained.

In §5, we shall consider submanifolds in R^N with parallel mean curvature, including minimal submanifolds and minimal real Kähler submanifolds. The so-called minimal real Kählerr submanifolds are generalizations of Kähler submanifolds.

From Theorem 3.4 and Example 3.4, we immediately get

Theorem 3.8. *Let (M^m, g, J) be a complex m -dimensional Kähler manifold which posses a complex 2-exhaustion function Φ . If M has constant scalar curvature, then*

$$\frac{1}{\rho_1^{\bar{\lambda}}} \int_{B_{\Phi}(\rho_1)} |\Omega_{ric}|^2 dv \leq \frac{1}{\rho_2^{\bar{\lambda}}} \int_{B_{\Phi}(\rho_2)} |\Omega_{ric}|^2 dv$$

for any $0 < \rho_1 \leq \rho_2$, where $\bar{\lambda} = \bar{k}_1/\bar{k}_2$.

By Example 2.4 and Theorem 3.8, we have

Corollary 3.1. *Let M^m ($m > 2$) be a complex m -dimensional Stein manifold with metric and Φ as in Example 2.4. If M has constant scalar curvature, then*

$$\frac{1}{\rho_1^{2(m-2)}} \int_{B_{\Phi}(\rho_1)} |\Omega_{ric}|^2 dv \leq \frac{1}{\rho_2^{2(m-2)}} \int_{B_{\Phi}(\rho_2)} |\Omega_{ric}|^2 dv$$

for any $0 < \rho_1 \leq \rho_2$.

Taking $p = 2$ in Lemma 2.4, and using Theorem 3.8, we yield

Corollary 3.2. *Let M^m ($\dim_{\mathbb{C}} M = m$) be a complete Kähler manifold with a pole. Assume that the radial curvature of M satisfies one of the following three conditions:*

- (i) $-\alpha^2 \leq K_r \leq -\beta^2$ with $\alpha > 0$, $\beta > 0$ and $m\beta - 2\alpha \geq 0$;
- (ii) $-\frac{A}{(1+r^2)^{1+\epsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\epsilon}}$ with $\epsilon > 0$, $A \geq 0$, $0 \leq B < 2\epsilon$ and $1 + (2m - 1)[1 - \frac{B}{2\epsilon}] - 4e^{A/2\epsilon} > 0$;
- (iii) $-\frac{a^2}{1+r^2} \leq K_r \leq \frac{b^2}{1+r^2}$ with $a \geq 0$, $b^2 \in [0, 1/4]$ and $2 + (2m - 1)[1 + \sqrt{1 - 4b^2}] - 4[1 + \sqrt{1 + 4a^2}] > 0$.

If M has constant scalar curvature, then

$$\frac{1}{\rho_1^{\bar{\lambda}}} \int_{B_{\Phi}(\rho_1)} |\Omega_{ric}|^2 dv \leq \frac{1}{\rho_2^{\bar{\lambda}}} \int_{B_{\Phi}(\rho_2)} |\Omega_{ric}|^2 dv$$

for any $0 < \rho_1 \leq \rho_2$, where

$$(3.22) \quad \bar{\lambda} = \begin{cases} 2[m - 2\frac{\alpha}{\beta}] & \text{if } K_r \text{ satisfies (i),} \\ 1 + (2m - 1)(1 - \frac{B}{2\epsilon}) - 4e^{A/2\epsilon} & \text{if } K_r \text{ satisfies (ii),} \\ 1 + \frac{(2m-1)[1+\sqrt{1-4b^2}]}{2} - 2[1 + \sqrt{1 + 4a^2}] & \text{if } K_r \text{ satisfies (iii).} \end{cases}$$

Now we deduce some vanishing theorems from the above monotonicity formulae. First, by Theorem 3.1 and Theorem 3.4, we have:

Corollary 3.3. *Let M^m , $\xi : E \rightarrow M$, Φ and ω be either as in Theorem 3.1 or as in Theorem 3.4. If ω either satisfies*

$$\int_{B_{\Phi}(R)} |\omega|^2 dv = o(R^{\lambda}) \quad \text{as } R \rightarrow \infty,$$

or

$$\int_{B_{\Phi}(R)} |\omega|^2 dv = o(R^{\bar{\lambda}}) \quad \text{as } R \rightarrow \infty,$$

where λ and $\bar{\lambda}$ are as in Theorem 3.1 and Theorem 3.4 respectively, then $\omega = 0$. In particular, if ω has finite energy, then $\omega = 0$.

By Theorem 3.2 and Theorem 3.5, we derive the following two corollaries:

Corollary 3.4. *Let (M, g) , $\xi : E \rightarrow M$, Φ , $B_\Phi(R_0)$ and $\omega \in A^p(\xi)$ be as in Theorem 3.2. Set $v = \frac{\nabla \Phi}{|\nabla \Phi|}$. If ω satisfies*

$$\frac{|\omega|^2}{2} \geq |i_v \omega|^2$$

on $\partial B_\Phi(R_0)$ and

$$\int_{B_\Phi(R) \setminus B_\Phi(R_0)} |\omega|^2 dv = o(R^{\lambda(R_0)}) \quad \text{as } R \rightarrow \infty,$$

then $\omega \equiv 0$ on $M \setminus B_\Phi(R_0)$. In particular, if ω has finite energy, then $\omega \equiv 0$ on $M \setminus B_\Phi(R_0)$.

Corollary 3.5. *Let (M, g) , $\xi : E \rightarrow M$, Φ , $B_\Phi(R_0)$ and $\omega \in A^1(\xi)$ be as in Theorem 3.5. If*

$$\int_{B_\Phi(R) \setminus B_\Phi(R_0)} |\omega|^2 dv = o(R^{\bar{\lambda}(R_0)}) \quad \text{as } R \rightarrow \infty,$$

then $\omega \equiv 0$ on $M \setminus B_\Phi(R_0)$. In particular, if ω has finite energy, then $\omega \equiv 0$ on $M \setminus B_\Phi(R_0)$.

Remark 3.1. *If ω in Corollary 3.4 or Corollary 3.5 posses the unique continuation property, then $\omega = 0$ on the whole M .*

It follows from Theorem 3.6 that

Corollary 3.6. *Let M^m be a complex m -dimensional Stein manifold with metric and Φ as in Theorem 3.6. Suppose $\xi : E \rightarrow M$ is a Riemannian vector bundle and $\omega \in A^p(\xi)$ with $p < m$ is J -invariant. If ω satisfies the conservation law and*

$$\int_{B_\Phi(\rho)} |\omega|^2 dv = o(\rho^{2(m-p)}) \quad \text{as } \rho \rightarrow \infty,$$

then $\omega = 0$.

From Theorem 3.3 and Theorem 3.7, we get

Corollary 3.7. *Let M^m , ξ , K_r and ω be as in Theorem 3.3 (resp. Theorem 3.7). Then $\omega = 0$ if ω satisfies*

$$\int_{B_\rho(x_0)} |\omega|^2 dv = o(\rho^\lambda) \quad (\text{resp. } o(\rho^{\bar{\lambda}})) \quad \text{as } \rho \rightarrow \infty,$$

where λ (resp. $\bar{\lambda}$) is given by (3.9) (resp. (3.18)).

By Corollary 3.1, we have

Corollary 3.8. *Let M^m ($m > 2$) be a complex m -dimensional Stein manifold with metric and Φ as in Example 2.4. If M has constant scalar curvature and*

$$\int_{B_\Phi(\rho)} |\Omega_{ric}|^2 dv = o(\rho^{2(m-2)}),$$

then M is holomorphically isometric to C^m .

Proof. Actually, in Example 2.4, M is embedded in some C^N as a closed complex submanifold and endowed with the induced metric. From the growth condition and Corollary 3.1, we derive that M is Ricci flat and thus its scalar curvature is zero. It is clear that the second fundamental form of M as a submanifold in C^N vanishes identically. Therefore M is holomorphically isometric to C^m . ■

By Corollary 3.2, we yield

Corollary 3.9. *Let M^m ($\dim_C M = m$) be a complete Kähler manifold with a pole. Assume that the radial curvature of M satisfies one of the following two conditions:*

- (i) $-\frac{A}{(1+r^2)^{1+\epsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\epsilon}}$ with $\epsilon > 0$, $A \geq 0$, $0 \leq B < 2\epsilon$ and $1 + (2m - 1)[1 - \frac{B}{2\epsilon}] - 4e^{A/2\epsilon} > 0$;
- (ii) $-\frac{a^2}{1+r^2} \leq K_r \leq \frac{b^2}{1+r^2}$ with $a \geq 0$, $b^2 \in [0, 1/4]$ and $2 + (2m - 1)[1 + \sqrt{1 - 4b^2}] - 4[1 + \sqrt{1 + 4a^2}] > 0$.

Then M is Ricci flat if it has constant scalar curvature and satisfies the following growth condition:

$$\int_{B_\Phi(\rho)} |\Omega_{ric}|^2 dv = o(\rho^{\bar{\lambda}}),$$

where

$$\bar{\lambda} = \begin{cases} 1 + (2m - 1)(1 - \frac{B}{2\epsilon}) - 4e^{A/2\epsilon} & \text{if } K_r \text{ satisfies (i),} \\ 1 + \frac{(2m-1)[1+\sqrt{1-4b^2}]}{2} - 2[1 + \sqrt{1 + 4a^2}] & \text{if } K_r \text{ satisfies (ii).} \end{cases}$$

Some Ricci flatness results were established in [28] for a complete ALE Kähler manifold by assuming nonpositivity or nonnegativity of Ricci curvature and a growth condition of the scalar curvature. On the other hand, if the curvature of a Kähler manifold decays like $r^{-2-\epsilon}$, the authors in [27, 28] could deduce, under some extra conditions, that M is isometrically biholomorphic to C^m . In following, we show that Corollary 3.9 can be used to establish such a kind of results.

Corollary 3.10. *Let M^m ($\dim_C M = m$) be a complete Kähler manifold whose sectional curvature dose not change sign. Suppose that M has a pole x_0 and the radial curvature of M satisfies one of the following two conditions*

- (i) $-\frac{a^2}{1+r^2} \leq K_r \leq 0$ with $a \geq 0$ and $m - [1 + \sqrt{1 + 4a^2}] > 0$;
- (ii) $0 \leq K_r \leq \frac{b^2}{1+r^2}$ with $b^2 \in [0, 1/4]$ and $(2m - 1)[1 + \sqrt{1 - 4b^2}] - 6 > 0$.

Then M is holomorphically isometric to C^m if it has constant scalar curvature and satisfies the following growth condition:

$$\int_{B_\Phi(\rho)} |\Omega_{ric}|^2 dv = o(\rho^{\bar{\lambda}}),$$

where

$$\bar{\lambda} = \begin{cases} 2m - 2[1 + \sqrt{1 + 4a^2}] & \text{if } K_r \text{ satisfies (i),} \\ \frac{(2m-1)[1 + \sqrt{1 - 4b^2}] - 6}{2} & \text{if } K_r \text{ satisfies (ii).} \end{cases}$$

Proof. As in Corollary 3.9, M is Ricci flat. Since the curvature of M does not change sign, we deduce that M is flat. It follows that M is holomorphically isometric to C^m . \blacksquare

4 Monotonicity Formulae for volumes

In this section, we establish monotonicity formulae for volumes, especially for the volumes of minimal submanifolds whose outer spaces poss some suitable exhaustion functions.

To estimate the volumes, we introduce the following notion. Let (N^n, g) be a Riemannian manifold and let Φ be a Lipschitz continuous function on N^n satisfying:

(4.1) $\Phi \geq 0$ and Φ is an exhaustion function of M ;

(4.2) $\Psi = \Phi^2$ is of class C^∞ and Ψ has only discrete critical points;

(4.3) Fix an integer $m \leq n$, the constant $\hat{k}_1 = \inf_{x \in M} \{\lambda_1(x) + \cdots + \lambda_m(x)\} \geq 0$,

where $\lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_n(x)$ are the eigenvalues of $Hess(\Psi)$, and the constant $k_2 = \sup_{x \in M} |\nabla \Phi|^2$ is finite. Set

$$(4.4) \quad \hat{\lambda} = \frac{\hat{k}_1}{2k_2}.$$

Notice that (4.3) implies in particular that Ψ is strictly m -subharmonic. If $m = n$, then Φ becomes a real 0-exhaustion function in the sense of §2.

Theorem 4.1. *Let (N^n, g) be a Riemannian manifold with an exhaustion function satisfying (4.1), (4.2) and (4.3). Let $B_\Phi(t) \subset N$ be the sublevel set of Φ . Suppose $i : M^m \rightarrow N^n$ is a minimal submanifold of dimension m , then*

$$\frac{Vol(M \cap B_\Phi(\rho_1))}{\rho_1^{\hat{\lambda}}} \leq \frac{Vol(M \cap B_\Phi(\rho_2))}{\rho_2^{\hat{\lambda}}}$$

for any $0 < \rho_1 \leq \rho_2$, where $\hat{\lambda}$ is given by (4.4).

Proof. Let $\rho : M \rightarrow R^+$ be the restriction of Φ to M . Denote by D and ∇ the covariant derivatives of N^n and M^m respectively. By the composition formula (cf. [16]), we have

$$\begin{aligned} \nabla d(\Phi^2 \circ i)(X, Y) &= (\nabla d\Phi^2)(di(X), di(Y)) + d\Phi^2[(\nabla di)(X, Y)] \\ (4.5) \qquad \qquad \qquad &= (\nabla d\Psi)(X, Y) + \langle A(X, Y), \nabla\Psi \rangle \end{aligned}$$

for any $X, Y \in TM$, where $A(X, Y) = (D_X Y)^\top$.

Take ρ as an exhaustion function on M . By definition of \hat{k}_1 and the minimality of M , we get from (4.5) that

$$\begin{aligned} k_1 &= \inf_{x \in M} \{\tilde{\lambda}_1(x) + \cdots + \tilde{\lambda}_m(x)\} \\ &\geq \inf_{x \in M} \{\lambda_1(x) + \cdots + \lambda_m(x)\} \\ &= \hat{k}_1, \end{aligned}$$

where $\tilde{\lambda}_1(x) \leq \tilde{\lambda}_2(x) \leq \cdots \leq \tilde{\lambda}_n(x)$ are the eigenvalues of $Hess(\rho^2)$. Thus k_1 is positive and ρ is a real 0-exhaustion function.

Now we choose $\omega = 1 \in A^0(M \times R)$ which clearly satisfies the conservation law by (2.3). From the proof of Theorem 3.1, we see that k_2 can be any positive number larger than $\sup_{x \in M} |\nabla \rho|^2$ and k_1 can be any positive number less than $\inf_{x \in M} \{\tilde{\lambda}_1(x) + \cdots + \tilde{\lambda}_m(x)\}$.

By definition of k_2 , we have

$$|\nabla \rho| = |\nabla \Phi| \leq k_2.$$

Hence the exhaustion function ρ has growth order $\hat{\lambda}$. Therefore Theorem 3.1 yields

$$\frac{Vol(M \cap B_\Phi(\rho_1))}{\rho_1^{\hat{\lambda}}} \leq \frac{Vol(M \cap B_\Phi(\rho_2))}{\rho_2^{\hat{\lambda}}}.$$

■

Remark 4.1. (1) From the proof of Theorem 5.1, we see that if $m - 2 - mp\delta > 0$ (resp. $m - 2 - 2p\delta > 0$) for a complete submanifold with parallel mean curvature (resp. for a complete minimal submanifold) in R^N , then ρ^2 is a real p -exhaustion function. (2) If we consider the identity immersion $x = I : N \rightarrow N$ in Theorem 4.1, then we get in particular the following

$$\frac{Vol(B_\Phi(\rho_1))}{\rho_1^{\hat{\lambda}}} \leq \frac{Vol(B_\Phi(\rho_2))}{\rho_2^{\hat{\lambda}}}$$

for any $0 < \rho_1 \leq \rho_2$.

Corollary 4.1. *Let (N^n, g) be a Riemannian manifold with an exhaustion function satisfying (4.1), (4.2) and (4.3) outside a sublevel set $B_\Phi(R_0)$ for some $R_0 > 0$. Suppose $M^m \rightarrow N^n$ is a minimal submanifold and $x_0 \in M$. Let $B_{x_0}(r) \subset N$ be the geodesic ball of radius r centered at x_0 . Then*

$$\frac{\text{Vol}(M \cap (B_{x_0}(\rho_1) - B_{x_0}(R_0)))}{\rho_1^{\hat{\lambda}}} \leq \frac{\text{Vol}(M \cap (B_{x_0}(\rho_2) - B_{x_0}(R_0)))}{\rho_2^{\hat{\lambda}}}$$

for any $0 < R_0 \leq \rho_1 < \rho_2$.

Proof. By the proof of Theorem 3.2 and Theorem 4.1, we can complete the proof of this corollary. ■

When choosing the distance function to be the exhaustion function, we can get the following lemma of growth order.

Lemma 4.1. *Let (N^n, g) be a complete Riemannian manifold with a pole x_0 and let r be the distance function relative to x_0 . Assume that the radial curvature K_r of N satisfies one of the following conditions:*

- (i) $K_r \leq -\beta^2$ with $\beta > 0$;
- (ii) $K_r \leq \frac{B}{(1+r^2)^{1+\epsilon}}$ with $\epsilon > 0$, $0 \leq B < 2\epsilon$;
- (iii) $K_r \leq \frac{b^2}{1+r^2}$ with $b^2 \in [0, 1/4]$.

Then r^2 is an exhaustion function satisfying (4.1), (4.2), (4.3) and

$$(4.6) \quad \hat{\lambda} = \begin{cases} 1 + (m-1)\beta r \coth(\beta r) & \text{if } K_r \text{ satisfies (i),} \\ m(1 - \frac{B}{2\epsilon}) & \text{if } K_r \text{ satisfies (ii),} \\ \frac{m(1+\sqrt{1-4b^2})}{2} & \text{if } K_r \text{ satisfies (iii).} \end{cases}$$

Proof. (i) By the Hessian comparison theorem

$$\begin{aligned} \text{Hess}(r^2) &= 2dr \otimes dr + 2r \text{Hess}(r) \\ &\geq 2dr \otimes dr + 2r\beta r \coth(\beta r)(g - dr \otimes dr) \end{aligned}$$

in the sense of quadratic forms and $\beta r \coth(\beta r) \geq 1$. Thus we have $\hat{k}_1 = \inf_{x \in M} \{\lambda_1(x) + \dots + \lambda_m(x)\} = 2 + 2(m-1)\beta r \coth(\beta r)$ and $k_2 = |\nabla r|^2 = 1$. Therefore $\hat{\lambda} = 1 + (m-1)\beta r \coth(\beta r)$.

The other cases can be proved similarly. ■

Theorem 4.2. *Let (N^n, g) be a complete Riemannian manifold with a pole x_0 and let r be the distance function relative to x_0 . Assume that the radial curvature K_r of N satisfies one of the following conditions:*

- (i) $K_r \leq -\beta^2$ with $\beta > 0$;
- (ii) $K_r \leq \frac{B}{(1+r^2)^{1+\epsilon}}$ with $\epsilon > 0$, $0 \leq B < 2\epsilon$;

(iii) $K_r \leq \frac{b^2}{1+r^2}$ with $b^2 \in [0, 1/4]$.

Suppose $i : M^m \rightarrow N^n$ is a minimal submanifold and $x_0 \in M$. Let $B_{x_0}(r) \subset N$ be the geodesic ball of radius r centered at x_0 . Then

$$\frac{\text{Vol}(M \cap B_{x_0}(\rho_1))}{\widehat{\rho_1^\lambda}} \leq \frac{\text{Vol}(M \cap B_{x_0}(\rho_2))}{\widehat{\rho_2^\lambda}},$$

for any $0 < \rho_1 < \rho_2$, where $\widehat{\lambda}$ is given by (4.6).

Proof. Let $\rho : M \rightarrow R^+$ be the extrinsic distance function of M relative to x_0 , that is, $\rho(x) = r \circ i(x)$ for $x \in M$. By Theorem 4.1 and Lemma 4.1, we can immediately get the theorem. ■

When outer space is Euclidean space, we can recapture the classical Monotonicity formulae of the volume of minimal submanifolds.

Corollary 4.2. (cf. [11]) Suppose that $M \subset R^n$ is a minimal submanifold and the origin $o \in M$. Let $B(r)$ be the Euclidean ball of radius r centered at o , then for all $0 < \rho_1 < \rho_2$

$$\frac{\text{Vol}(M \cap B(\rho_1))}{\rho_1^m} \leq \frac{\text{Vol}(M \cap B(\rho_2))}{\rho_2^m}$$

with equality if and only if M is an affine m -plane.

Proof. By choosing $B = 0$ in Theorem 4.2 (ii), the above inequality can be obtained directly.

For the second conclusion, checking the procedure of the proof of Theorem 3.1, the above equality holds if and only if

$$|\nabla r(x)| = |Dr(x)| = k_2 = 1 \quad \text{for any } x \in M,$$

that is

$$\langle x, \eta \rangle = 0$$

for any $\eta \in T_x^\perp M$. For $x \neq 0$, we defined

$$e_x = \frac{x}{|x|} \in T_x M.$$

Then we have a unit vector field e on $M \setminus \{o\}$. Obviously the integral curves of e are straight lines in R^n passing through o , because $D_e e = 0$. We conclude that M is an m -plane passing through o . ■

5 Application to Bernstein type problem

In this section, we will investigate submanifolds in a Euclidean space with parallel mean curvature. It is known that the Gauss maps of such submanifolds are harmonic.

It is natural to apply the method developed in previous sections to establish Liouville theorems of the Gauss map, which are equivalent to Bernstein type results of the submanifolds. Throughout this section, we shall make use of the following notations:

ρ = the extrinsic distance relative to a point $o \in M$,

$D(R)$ = the intersection of M with the Euclidean R -ball centered at $o \in M$.

5.1. Submanifolds in Euclidean spaces

Theorem 5.1. *Let $M^m(m \geq 3)$ be a complete Riemannian manifold and let $x : M \rightarrow R^N$ be a proper immersion with parallel mean curvature. Assume that $\delta = \sup_{x \in M} \rho(x) \|A\|_x$ is finite, where A is the second fundamental form of M .*

(i) *If the constant $\mu_1 = m - 2 - m\delta > 0$ and*

$$(5.1) \quad \int_{D(\rho)} |A|^2 dv = o(\rho^{\mu_1}) \quad \text{as } \rho \rightarrow \infty,$$

then M is an affine m -plane in R^N .

(ii) *If M is minimal, the constant $\mu_2 = m - 2 - 2\delta > 0$ and*

$$(5.2) \quad \int_{D(\rho)} |A|^2 dv = o(\rho^{\mu_2}) \quad \text{as } \rho \rightarrow \infty,$$

then M is an affine m -plane in R^N .

Proof. As in the proof of Theorem 4.1, we take $\Phi = \rho$ as an exhaustion function for M . We need to prove that Φ is a real 1-exhaustion function under the assumptions. Let $\lambda_1(x) \leq \lambda_2(x) \leq \dots \leq \lambda_m(x)$ be the eigenvalues of $Hess(\rho^2)$. From (4.5), we get

$$(5.3) \quad Hess(\rho^2)(X, X) = 2 \langle X, X \rangle + 2 \langle A(X, X), X \rangle.$$

It follows that

$$2 - 2\delta \leq \lambda_i \leq 2 + 2\delta, \quad i = 1, \dots, m.$$

Hence

$$(5.4) \quad \begin{aligned} k_1 &= \inf_{x \in M} \left\{ \sum_{i=1}^m \lambda_i(x) - 2\lambda_m(x) \right\} \\ &\geq 2m - 4 - 2m\delta. \end{aligned}$$

By definition of λ and $|\nabla \rho| \leq 1$, we have from (2.11) and (5.4) that

$$\lambda \geq \frac{k_1}{2} = m - 2 - m\delta.$$

Since M has parallel mean curvature, its Gauss map $\gamma : M \rightarrow G_m(R^N)$ is harmonic. Thus $d\gamma$ satisfies the conservation law, i.e., $div S_\gamma = 0$. From [16], we know that

$|d\gamma|^2 = \|A\|^2$. By applying Corollary 3.3 to $d\gamma$ under the assumption (5.1), we deduce $d\gamma = 0$, that is γ is constant. Hence M is an m -plane.

Now suppose that M is minimal. By (5.3) and the minimality of M , we deduce

$$\begin{aligned} k_1 &= \inf_{x \in M} \left\{ \sum_{i=1}^m \lambda_i(x) - 2\lambda_m(x) \right\} \\ &\geq 2m - 4 - 4\delta. \end{aligned}$$

So $\lambda \geq m - 2 - 2\delta$. Therefore a similar argument yields the result of (ii). ■

Remark 5.1. (1) Actually we may get monotonicity formulae for the integral of $\|A\|^2$ in Theorem 5.1. (2) In the case that M is minimal, Chen [8] proved that if $\delta < 1$ (without any energy growth condition), then M is an m -plane in R^N . (3) Kasue [21] proved that if $\|A\|^2 = o(\frac{1}{\rho^{2+\epsilon}})$ (as $\rho \rightarrow \infty$) and $M^m(m \geq 3)$ has one end, then M is an m -plane.

Corollary 5.1. Let $M^m(m \geq 3)$ be a complete Riemannian manifold and let $x : M^m \rightarrow R^N$ be a proper immersion with parallel mean curvature (resp. $H = 0$). Let δ , A , ρ and μ_i ($i=1,2$) be as in Theorem 5.1. If μ_1 (resp. $\mu_2 > 0$) and

$$\int_M \|A\|^2 dv < +\infty,$$

then M is an affine m -plane in R^N .

Corollary 5.2. Let $x : M^m \rightarrow R^{m+1}(m \geq 3)$ be a properly immersed complete hypersurface. Set $\tilde{\delta} = \sup_{x \in M} \{\rho(x) \max_i |\eta_i(x)|\}$, where $\eta_i(x)(i = 1, \dots, m)$ are the principal curvatures of M at x . Suppose M satisfies one of the following two conditions:

(i) M has parallel mean curvature, the constant $\tilde{\mu}_1 = m - 2 - m\tilde{\delta} > 0$ and

$$\int_{D(\rho)} \|A\|^2 dv = o(\rho^{\tilde{\mu}_1}) \quad \text{as } \rho \rightarrow \infty,$$

(ii) M is minimal, the constant $\tilde{\mu}_2 = m - 2 - 2\tilde{\delta} > 0$ and

$$\int_{D(\rho)} \|A\|^2 dv = o(\rho^{\tilde{\mu}_2}) \quad \text{as } \rho \rightarrow \infty,$$

Then M is a hyperplane.

Proof. Let \vec{n} be the unit normal vector field of M in R^{m+1} . From (5.3), we get

$$\lambda_i(x) = 2 + 2\eta_i(x) < \vec{n}, x >.$$

Therefore

$$2 - 2\tilde{\delta} \leq \lambda_i(x) \leq 2 + 2\tilde{\delta}.$$

So

$$k_1 = \inf_{x \in M} \left\{ \sum_{i=1}^m \lambda_i(x) - 2\lambda_m(x) \right\} \geq 2m - 4 - 2m\tilde{\delta}.$$

In particular, if M is minimal, then

$$k_1 \geq 2m - 4 - 4\tilde{\delta}.$$

The remaining arguments are similar to those in the proof of Theorem 5.1. ■

Theorem 5.2. *Let $x : M^m \rightarrow R^N$ be a properly immersed m -dimensional ($m \geq 3$) complete submanifold. Set $\delta(R_0) = \sup_{M \setminus B(R_0)} \{\rho(x) \|A\|(x)\}$ for some regular value $R_0 >$*

0 of ρ and $\nu = \frac{\nabla \rho}{|\nabla \rho|}$.

(i) Suppose M has parallel mean curvature. Suppose $\|i_\nu A\|^2 \leq \frac{\|A\|^2}{2}$ on $\partial D(R_0)$ and $\|A\|(x) \leq \frac{m-2-\epsilon_0}{m\rho(x)}$ for some $\epsilon_0 > 0$ and any $\rho(x) \geq R_0$. If

$$(5.5) \quad \int_{D(\rho)} |A|^2 dv = o(\rho^{\mu_1}) \quad \text{as } \rho \rightarrow \infty,$$

where $\mu_1 = m - 2 - m\delta(R_0)$, then M is an m -plane in R^N .

(ii) Suppose M is a minimal submanifold. Suppose $\|i_\nu A\|^2 \leq \frac{\|A\|^2}{2}$ on $\partial D(R_0)$ and $\|A\|(x) \leq \frac{m-2-\epsilon_0}{2\rho(x)}$ for some $\epsilon_0 > 0$ and any $\rho(x) \geq R_0$. If

$$(5.6) \quad \int_{D(\rho)} |A|^2 dv = o(\rho^{\mu_2}) \quad \text{as } \rho \rightarrow \infty,$$

where $\mu_2 = m - 2 - 2\delta(R_0)$, then M is an m -plane in R^N .

Proof. First, we assume that M has parallel mean curvature. Set

$$\begin{aligned} k_1(R_0) &= \inf_{M \setminus B(R_0)} \left\{ \sum_{i=1}^m \lambda_i(x) - 2\lambda_m(x) \right\}, \\ k_2(R_0) &= \sup_{M \setminus B(R_0)} |\nabla \rho|^2, \\ \lambda(R_0) &= \frac{k_1(R_0)}{2k_2(R_0)}, \end{aligned}$$

where $\lambda_1(x) \leq \lambda_2(x) \leq \dots \leq \lambda_m(x)$ are the eigenvalues of $Hess(\rho^2)$. From the proof of Theorem 5.1, it is easy to see that

$$\lambda(R_0) \geq m - 2 - m\delta(R_0).$$

If $\|A\| \leq \frac{m-2-\epsilon_0}{m\rho(x)}$ for some $\epsilon_0 > 0$ and any $\rho(x) \geq R_0$, then

$$m - 2 - m\delta(R_0) \geq \epsilon_0 > 0.$$

Obviously the assumption $\|i_\nu A\|^2 \leq \frac{\|A\|^2}{2}$ is equivalent to $\|d\gamma(\nu)\|^2 \leq \frac{\|d\gamma\|^2}{2}$, where $\gamma : M \rightarrow G(m, N)$ is the Gauss map. We have already known that γ is harmonic. We deduce from Corollary 3.4 that γ is constant on $M \setminus B(R_0)$ and thus γ is constant on M by the unique continuation property of harmonic maps. Hence M is an m -plane. ■

Corollary 5.3. *Let $x : M^m \rightarrow R^N$ be an m -dimensional ($m \geq 3$) complete submanifold with parallel mean curvature. Suppose $|i_v A|^2 \leq \frac{|A|^2}{2}$, where $v = \frac{\nabla \rho}{|\nabla \rho|}$. If*

$$(5.7) \quad \|A\|(x) = o\left(\frac{1}{\rho(x)}\right) \quad \text{as } \rho(x) \rightarrow +\infty,$$

and

$$\int_M |A|^2 dv < +\infty,$$

then M is an affine m -plane in R^N .

Proof. By (5.7) and Theorem 1.1 of [6], we know that $x : M^m \rightarrow R^N$ is proper. So the extrinsic distance function ρ is an exhaustion function. The remaining arguments are as above. ■

Remark 5.2. *Some submanifolds automatically satisfy the condition*

$$(5.8) \quad |i_v A|^2 \leq \frac{|A|^2}{2}$$

in Theorem 5.2 or Corollary 5.3. Recall that the submanifolds M with the second fundamental form A is called isotropic at $p \in M$ if $A(e, e)$ is independent of the choice of the unit vector $e \in T_p M$. M is said to be isotropic if M is isotropic at every point. It is easy to verify that an isotropic submanifold with dimension ≥ 2 satisfies (5.8). In section 5.2, we will investigate minimal real Kähler submanifolds in R^N , which automatically satisfy (5.8) too.

Definition 5.1. (cf. [1]) *The integral $\int_M \|A\|^m dv$ for m -dimensional minimal submanifold M in R^N is called the total scalar curvature of M .*

Lemma 5.1. *Let $M^m \rightarrow R^{m+p}$ ($m \geq 3$) be a minimally immersed complete submanifold. If M has finite total scalar curvature, then*

- (1) $\|A\|_x \leq \frac{C}{\rho^m}$ for $x \in \partial D(\rho)$;
- (2) $\int_M \|A\|^2 dv < \infty$.

Proof. (1) See [1, 34, 40].

(2) By using (1), the proof goes almost the same as that of [36], except that one considers a minimal submanifold instead of a minimal hypersurface. Since the proof has no essential difference, we omit the detailed proof about (2) here. ■

Remark 5.3. *In [36], the authors showed the finiteness of $\|A\|_{L^2(M)}$ for minimal hypersurfaces with finite total scalar curvature.*

Corollary 5.4. *Let $x : M^m \rightarrow R^N$ ($m \geq 3$) be an m -dimensional complete minimal submanifold in R^N with finite total scalar curvature. If $|i_v A|^2 \leq \frac{|A|^2}{2}$ holds outside a compact subset of M , then M is an affine m -plane in R^N .*

Proof. From Lemma 2.4 of [1], we know that $x : M^m \rightarrow R^N$ is proper. So the extrinsic distance function ρ is an exhaustion function. The Corollary 5.4 follows immediately from Lemma 5.1 and Corollary 5.3. ■

Corollary 5.5. *Let $x : M^m \rightarrow R^{m+1}$ ($m \geq 3$) be a minimally immersed complete hypersurface with finite total scalar curvature. Let $\eta_i(x)$ ($i = 1, \dots, m$) be the principal curvatures of M at x . If there exists a compact subset K of M such that*

$$(5.9) \quad \max_i \{\eta_i^2(x)\} \leq \frac{1}{2} \|A\|_x^2$$

for any $x \in M \setminus K$, then M is an affine hyperplane in R^{m+1} .

Proof. Clearly the result of this corollary follows immediately from Corollary 5.4. ■

Remark 5.4. (1) Suppose $\mu_2(R_0)$ is as in Theorem 5.2 and R_0 is sufficiently large so that $\mu_2(R_0) > 0$. To get the result of Corollary 5.5, we only need to assume (5.8) holds on $\partial D(R_0)$; (2) It is known that the principal curvatures of the catenoid in R^{m+1} are (cf. [41]) $\eta_2 = \eta_3 = \dots = \eta_m = \eta$, $\eta_1 = -(m-1)\eta$. Clearly the catenoid does not satisfy the condition (5.8) for $m \geq 3$. This shows that the condition (5.8) (at least on $\partial D(R_0)$) is necessary for the result. (3) The known nontrivial examples of minimal submanifolds in R^N with finite total scalar curvature include catenoid (cf. [41]), and the Lagrangian catenoid [7] and the special Lagrangian submanifolds constructed by Lawlor [23]. (4) For a minimal submanifold $M^m \subset R^N$ with finite total scalar curvature, some authors proved the Bernstein type results under other geometric conditions, such as the stability ([36, 42]), having one end ([1]), monotonicity ([25]), etc. (5) In [10], the authors proved that if M is a submanifold with parallel mean curvature and $\int_M \|A\|^m dv < +\infty$, then M is minimal. Hence the results in Corollary 5.4 and Corollary 5.5 actually hold for submanifolds with parallel mean curvature.

5.2. The case of real Kähler submanifolds

Let M^m be a Kähler manifold of complex dimension m and let $x : M^m \rightarrow R^N$ an isometric immersion. We will call M a real Kähler submanifold in R^N . Considering the complexified cotangent bundle of M , with the usual procedure, its second fundamental form A can be split into different components as $A^{(p,q)}$, the (p, q) component ($p + q = 2, 0 \leq p, q \leq 2$). A real Kähler submanifold M is said to be (p, q) -geodesic if $A^{(p,q)} = 0$.

Notice that $(1, 1)$ -geodesic submanifolds have various different names (pluriminimal map, circular immersion) in literature (cf. [5, 14]). From [14], we know that $(1, 1)$ -geodesic submanifolds coincide with minimal real Kähler submanifolds. The simplest examples of pluriminimal submanifolds are minimal surfaces in R^N and Kähler submanifolds in C^m . In [2, 12, 19], the authors established Weierstrass representation for minimal real Kähler submanifolds. Therefore general minimal real

Kähler submanifolds exist in abundance in R^N too, besides the previous two special classes.

For a real Kähler submanifold $x : M^m \rightarrow R^N$, we have two Gauss maps: the usual Gauss map $\gamma : M \rightarrow G_{2m}(R^N)$ and the complex Gauss map $\gamma^C : M \rightarrow G_m(C^N)$. In [33], the authors defined the complex Gauss map

$$(5.10) \quad \gamma^C : M \rightarrow G_m(C^N)$$

by assigning to each point $p \in M$ the parallel translation in C^N of the complex m -space $dx(T_p M^{(0,1)})$ to the origin of C^N .

Choosing a local orthonormal frame $\{e_A\}_{A=1}^{2m} = \{e_1, \dots, e_m, Je_1, \dots, Je_m\}$ around p such that $(\nabla_{e_A} e_B)_p = 0$, we have local unitary frame fields $\{\eta_j\}_{j=1}^m \in T^{1,0}M$ and $\{\eta_{\bar{j}}\}_{j=1}^m \in T^{0,1}M$ given by:

$$(5.11) \quad \eta_j = \frac{1}{\sqrt{2}}(e_j - iJe_j), \quad \eta_{\bar{j}} = \frac{1}{\sqrt{2}}(e_j + iJe_j).$$

The complex Gauss map may be expressed by

$$(5.12) \quad \gamma^C = \eta_{\bar{1}} \wedge \dots \wedge \eta_{\bar{m}}.$$

Then we get

$$(5.13) \quad (d\gamma^C)_p(X) = \sum_{k=1}^m \eta_{\bar{1}} \wedge \dots \wedge A(X, \eta_{\bar{k}}) \wedge \dots \wedge \eta_{\bar{m}}.$$

Lemma 5.2. *Suppose $x : M^m \rightarrow R^N$ ($\dim_C M = m$) is a minimal real Kähler submanifold. Then γ^C is anti-holomorphic and $\|d\gamma^C\|^2 = \|A\|^2$.*

Proof. By the assumption that M is $(1,1)$ -geodesic, that is, $A(\eta_j, \eta_{\bar{k}}) = 0$, we get from (5.13) that

$$(d\gamma^C)_p(\eta_j) = 0, \quad j = 1, \dots, m.$$

Therefore the partial differential $\partial\gamma^C : T^{1,0}M \rightarrow T^{1,0}G_m(C^N)$ vanishes. This proves the anti-holomorphicity of γ^C . Clearly (5.13) gives

$$\|d\gamma^C\|^2 = 2 \sum_{j,k=1}^m \|A(\eta_{\bar{j}}, \eta_{\bar{k}})\|^2 = \|A\|^2.$$

■

Remark 5.5. *The authors in [33] established the anti-holomorphicity of γ^C for minimal real Kähler submanifolds. Let $H_m(C^N)$ be the space of complex m -dimensional isotropic subspaces of C^N . Actually the isometric immersion x factors through $H_m(C^N) \subset G_m(C^N)$ and thus γ^C becomes an anti-holomorphic map into $H_m(C^N)$. Therefore their result generalized Chern's result [9] about the Gauss map of minimal surfaces in R^N .*

Theorem 5.3. *Let $x : M^m \rightarrow R^N (m \geq 2)$ be a complete minimal real Kähler submanifold with complex dimension m . If*

$$\|A\|(x) = o\left(\frac{1}{\rho(x)}\right) \quad \text{as } \rho(x) \rightarrow +\infty$$

and

$$\int_M |A|^2 dv < +\infty,$$

then M is an affine $2m$ -plane in R^N .

Proof. Clearly γ^C as a special harmonic map satisfies the conservation law, that is, $\text{div} S_{\gamma^C} = 0$. Let ν be as in Theorem 5.2 and Corollary 5.4. By Lemma 5.2, γ^C is anti-holomorphic, so $d\gamma^C$ is pluriconformal. Therefore we have

$$\begin{aligned} \frac{|d\gamma^C|^2}{2} &\geq (d\gamma^C \odot d\gamma^C)(\nu, \nu) \\ &= |i_\nu d\gamma^C|^2 \end{aligned}$$

as shown in the proof of Theorem 3.4. Hence the result of Theorem 5.3 follows immediately from Corollary 5.3. ■

By Lemma 5.1 and Theorem 5.3, we can obtain the following theorem.

Theorem 5.4. *Let $x : M^m \rightarrow R^N (m \geq 2)$ be a complete minimal real Kähler submanifold with complex dimension m . If M has finite total scalar curvature, then M is an affine $2m$ -plane in R^N .*

Corollary 5.6. ([25]) *Let $M^m \rightarrow C^n (m \geq 2)$ be a complete Kähler submanifold with complex dimension m . If M has finite total scalar curvature, then M is an affine complex m -plane.*

Remark 5.6. *The assumption $m \geq 2$ in Theorem 5.4 or Corollary 5.6 is necessary, because there are many nontrivial examples of minimal surfaces in R^N with finite total curvature (cf. [29]).*

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